AN ARITHMETIC STUDY OF THE FORMAL LAPLACE TRANSFORM IN SEVERAL VARIABLES.

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Abstract. Let K be a number field, and let $K(x_1, ..., x_d)$ be the field of rational fractions in the variables $x_1, ..., x_d$. In this paper, we introduce two kinds of Laplace transform adapted to solutions of the differential $K(x_1, ..., x_d)$-modules with regular singularities, and give some of their basic differential and arithmetic properties. The purpose of this article is to provide some tools which might be useful, in particular, for the arithmetic study of the differential $K(x_1, ..., x_d)$-modules associated to E-functions in several variables.

1. Introduction

A power series $\sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$ is said to be *Gevrey of order s* if there exists a positive constant C such $|a_n| < C^n n!^s$ for all $n > 0$. It is known since the beginning of the last century that any formal power series arising in the asymptotic expansion of any solution of a linear or non-linear analytic differential equation is Gevrey of rational order s ([15], [19] and [22]). Moreover, the Gevrey series occurring in Taylor or asymptotic expansions of classical special functions have particular arithmetic properties which can be summarized in the concept of the *Gevrey series of arithmetic type* ([2], [4]):

A power series $\sum_{n\geq 0} a_n x^n$ is said to be *arithmetic Gevrey series* of order $s \in \mathbb{Q}$ if its coefficients a_n are algebraic numbers and if there exists a constant $C > 0$ such that the absolute value of the conjugates of the algebraic number $a_n/(n!)^s$ is less than C^n, and that, for all $n \in \mathbb{N}$, the common denominator d_n of the numbers $a_0 = a_0/(0!)^s, \ldots, a_n/(n!)^s$ is less than C^n. For instance, the Taylor expansion at the origin of the Airy function is arithmetic Gevrey series of precise order $-2/3$. This class includes the confluent and non confluent generalized hypergeometric series with rational parameters, the *Barnes* generalized hypergeometric series ${}_pF_{q-1}$ with rational parameters, any series which is algebraic over $\overline{\mathbb{Q}}(x)$, and two especially well-known series: *G-functions* $(s = 0)$ and *E-functions* $(s = -1)$ which have proved useful in number theory and have applications in transcendence proofs and differential equations (e.g. see [8], [7] and [18]).

The theory of G-functions (the case of arithmetic Gevrey series of order 0) in one-variable is now well known thanks to Bombieri, Chudnovsky, Dwork, André and others who, furthermore, brought to light its connections with arithmetic geometry (e.g. see [1], [9], [14] and [12]).

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A theory of arithmetic Gevrey series of order $s \neq 0$ in one-variable was recently developed by André in which a *Laplace transform* is used in an essential way [2]. Based on a theorem of Chudnovsky [12], he gave interesting structure results for the $\mathbb{C}[x]$-differential equation of minimal order annihilating an arithmetic Gevrey series of precise order $s \neq 0$. As an application, these results enabled him to deduce the fundamental theorem of the Siegel-Shidlovskii theory on the algebraic independence of values of E-functions at algebraic points, and the Lindemann-Weierstrass theorem.

A theory of G-functions in several variables was established in full generality by André, Baldassarri and Di Vizio ([5], [6] and [14]), but a more general theory of arithmetic Gevrey series in several variables has yet to mature. Given the work of André, it is natural that an appropriate notion of Laplace transform in several variables will be needed to build such theory.

In the present article, we introduce two kinds of Laplace transform which extend the ones introduced respectively in ([2], [20]) and [21] to fundamental solution matrices of integrable systems of partial differential equations on $K((x_1, \dots, x_d))$ with regular singularities at the origin (K being a number field). The general form of such fundamental solution matrices is $Yx_1^{\Lambda_1} \dots x_d^{\Lambda_d}$, where $\Lambda_1, \dots, \Lambda_d$ are square matrices with entries in the algebraic closure $\overline{K}$ of K, and where Y is an invertible matrix with entries in $\overline{K}((x_1, \dots, x_d))$ [16]. The new Laplace transforms studied in this paper are:

(1) the *standard Laplace transform*, which applies to the entries of $Yx_1^{\Lambda_1} \dots x_d^{\Lambda_d}$,
(2) the *formal Laplace transform*, which applies directly to the solution matrix $Yx_1^{\Lambda_1} \dots x_d^{\Lambda_d}$.

These transformations are given under the assumption that all the eigenvalues of $\Lambda_1, \dots,$ Λ_d are non-integral complex numbers, and have properties of commutations with the derivations $\partial/\partial x_i$ ($i = 1, \dots, d$) which extend those in one-variable case ((3.11) and (4.14)). Hence, they preserve the classical duality between the Laplace transform and the Fourier-Laplace transform. Moreover, for any $\underline{\tau} = (\tau_1, \dots, \tau_d) \in (K \setminus \{0\})^d$, the formal transformation is adapted to have a duality with the generalized Fourier-Laplace transform $\mathcal{F}_{\underline{\tau}}$ with respect to $\underline{\tau}$ (formula (4.14)), defined as the K-automorphism of $K[x_1, \dots, x_d, \partial/\partial x_1, \dots, \partial/\partial x_d]$ determined by:

$$\mathcal{F}_{\underline{\tau}}(x_i) = -\frac{1}{\tau_i}\frac{\partial}{\partial x_i}, \quad \mathcal{F}_{\underline{\tau}}(\frac{\partial}{\partial x_i}) = \tau_i x_i, \quad (i = 1, \dots, d).$$

For $\underline{\tau} = (1, \dots, 1)$, $\mathcal{F}_{\underline{\tau}}$ is just the classical Fourier-Laplace transform. In this case, we write $\mathcal{F}$ instead of $\mathcal{F}_{\underline{\tau}}$.

The difference between the two transformations is that the standard one involves transcendental numbers and applies to terms with logarithms (3.12), while the formal one is defined independently of such numbers and does not apply directly to terms with logarithms (4.13).

If v is a finite place of K above a prime number $p(v)$, we prove some arithmetic properties of the standard (resp. formal) Laplace transform in the case where all the eigenvalues of $\Lambda_1, \ldots, \Lambda_d$ belong to $K \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$ (resp. $\mathbb{Q} \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$) (Propositions 3.7 and 4.7).

As an application, we show in §3.4 how the standard Laplace transform acts on the arithmetic Gevrey series in several variables (Proposition 3.8).

We think that this paper provides some tools which might be useful for the development of an arithmetic theory of differential equations in higher dimension, and in particular, for the arithmetic study of the differential $K(x_1, \ldots, x_d)$-module generated by the different derivatives of a Gevrey series of nonzero order in several variables.

The aim of this theory is to develop techniques which allow, in particular, to obtain results on algebraic independence of values of the exponential function in several variables, and of the functions of the form:

$$f(x,y) = P(e^x,\ {}_2F_1(a,b,c;y)), \quad \text{with} \quad a,b,c \in \mathbb{Q},$$

where P is a polynomial. The values of such function are related to e and π.

2. Notation

Let K be a number field and let Σ_{f} be the set of all finite places v of K. For each $v \in \Sigma_{\mathrm{f}}$ above a prime number $p = p(v)$, we normalize the corresponding v-adic absolute value so that $|p|_v = p^{-1}$ and we put $\pi_v = |p|_v^{1/(p(v)-1)}$. We also fix an embedding $K \hookrightarrow \mathbb{C}$.

Let $K(x_1, \ldots, x_d)$ be the field of rational functions in the variables $x_1, \ldots, x_d$ with coefficients in K, with $d \in \mathbb{Z}_{>0}$. Put $\underline{x} = (x_1, \ldots, x_d)$, $\frac{1}{\underline{x}} = (\frac{1}{x_1}, \ldots, \frac{1}{x_d})$, $\partial_i = \frac{\partial}{\partial x_i}$, for $i = 1, \ldots, d$, $\underline{1} = (1, \ldots, 1)$, and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ for the elements of $\mathbb{N}^d$. For $\underline{\alpha}, \underline{\beta} \in \mathbb{N}^d$, $\gamma \in K$ and $n \in \mathbb{Z}_{>0}$, we set : $(\gamma)_0 = \gamma$, $(\gamma)_{n+1} = \gamma(\gamma+1)\ldots(\gamma+n)$,

$$|\underline{\alpha}| = \sum_{1 \le i \le d} \alpha_i, \quad \underline{\alpha}! = \prod_{1 \le i \le d} \alpha_i!, \quad \underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \ldots x_d^{\alpha_d}, \quad \underline{\partial}^{\underline{\alpha}} = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d},$$

$$\underline{\alpha} \le \underline{\beta} \iff \alpha_i \le \beta_i \quad \text{for all} \quad i = 1, \ldots, d, \quad \text{and} \quad \binom{\underline{\alpha}}{\underline{\beta}} = \prod_{1 \le i \le d} \binom{\alpha_i}{\beta_i} \quad \text{for} \quad \underline{\alpha} \ge \underline{\beta},$$

$$\underline{\alpha} < \underline{\beta} \iff \underline{\alpha} \le \underline{\beta} \quad \text{and} \quad \alpha_i < \beta_i \quad \text{for some} \quad i = 1, \ldots, d.$$

For a power series $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K[[\underline{x}]]$, we denote $f(\frac{1}{\underline{x}}) = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{-\underline{\alpha}} \in K[[\frac{1}{\underline{x}}]]$. If v is a finite place of Σ_{f} and $s \in \mathbb{Q}$ is a rational number, we set

$$\mathcal{R}_v^s(f) = \{ y \in K[[\underline{x}]] \mid r_v(y) \ge r_v(f)\ \pi_v^s \},$$

where $r_v(f)$ and $r_v(y)$ denote respectively the radius of convergence of f and y with respect to v. $\underline{\gamma} = (\gamma_1, \ldots, \gamma_d)$ and $\underline{k} = (k_1, \ldots, k_d)$ will denote respectively multi-exponents in K^d and $\mathbb{N}^d$. Also, we will use the following notation

$$(\log \underline{x})^{\underline{k}} = (\log x_1)^{k_1} \ldots (\log x_d)^{k_d}, \quad h_{\underline{\gamma},\underline{k}} = \underline{x}^{\underline{\gamma}} (\log \underline{x})^{\underline{k}},$$

$$(\underline{\gamma})_{\underline{0}} = \prod_{1\le i\le d} \gamma_i, \quad \text{and} \quad (\underline{\gamma})_{\underline{\alpha}} = (\gamma_1)_{\alpha_1}\dots(\gamma_d)_{\alpha_d}.$$

3. Standard Laplace transform

3.1 Review of the one-variable case. In [2], André extended the definition of the classical Laplace transform $\mathcal{L}$, in the formal way, to logarithmic solutions in one-variable. Later, in [20], the present author gave some arithmetic properties of this $\mathcal{L}$. In this paragraph, we recall the definition of the Laplace transform $\mathcal{L}(h_{\gamma,k})$ of the term $h_{\gamma,k} := x^\gamma(\log x)^k$ where $(\gamma,k)\in K\setminus\mathbb{Z}_{\le 0}\times\mathbb{N}$, and its differential and arithmetic properties [20, 4] [1].

Fix an embedding $K\hookrightarrow\mathbb{C}$. Let γ be an element of $K\setminus\mathbb{Z}_{\le 0}$ such that $\Re e(\gamma) > -1$, k a nonnegative integer, and let $h_{\gamma,k}$ denote the function defined by $h_{\gamma,k}(x) = x^\gamma(\log x)^k$; $x>0$. The classical Laplace transform of $h_{\gamma,0}$, denoted $\mathcal{L}(h_{\gamma,0})$, is defined by

$$\mathcal{L}(h_{\gamma,0})(x) = \int_0^\infty e^{-xt}h_{\gamma,0}(t)dt = \int_0^\infty e^{-xt}t^\gamma dt = \Gamma(\gamma+1)x^{-\gamma-1}.$$

The k-th derivative of this equation with respect to γ gives

$$(\frac{d}{d\gamma})^k\Big(\Gamma(\gamma+1)x^{-\gamma-1}\Big) = \int_0^\infty e^{-xt}t^\gamma(\log t)^k dt = \int_0^\infty e^{-xt}h_{\gamma,k}(t)dt = \mathcal{L}(h_{\gamma,k})(x),$$

and by iteration of Leibniz formula we find

$$\mathcal{L}(h_{\gamma,k})(x) = \sum_{j=0}^{k}\binom{k}{j}\Gamma^{(j)}(\gamma+1)x^{-\gamma-1}(-1)^{k-j}(\log x)^{k-j}, \tag{3.1}$$

where, $\Gamma^{(j)}$ denotes the j-th derivative of the Euler's Gamma function Γ [20, (4.2)]. To extend the definition of the Laplace transform $\mathcal{L}$ of $h_{\gamma,k}$ to any $\gamma\in K\setminus\mathbb{Z}_{\le 0}$, we have to introduce the functions $F_{\gamma,k,n}$ [20, (4.8)] [13, 5], defined for $n\in\mathbb{Z}_{>0}$ and $\gamma\in K\setminus\mathbb{Z}_{\le 0}$, by

$$F_{\gamma,k,n}(x) = \sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\frac{x^{\gamma+n+1}}{m+\gamma+1}\sum_{\ell=0}^{k}\frac{k!(-1)^{k-\ell}}{\ell!(m+\gamma+1)^{k-\ell}}(\log x)^\ell.$$

André observed that the function $x^{n+1}\mathcal{L}(F_{\gamma,k,n})$ is independent of the choice of n for $n\ge -\Re e(\gamma)-1$ (cf. [2, 5.3.6]). According to this remark, the Laplace transform $\mathcal{L}$, just defined for $\gamma\in K\setminus\mathbb{Z}_{\le 0}$ with $\Re e(\gamma)>-1$ (see (3.1) above), can be extended to any $\gamma\in K\setminus\mathbb{Z}_{\le 0}$ by putting

$$\mathcal{L}(h_{\gamma,k}) \;=\; x^{n+1}\mathcal{L}(F_{\gamma,k,n}), \quad \text{for} \quad n\ge -\Re e(\gamma)-1. \tag{3.2}$$

[1] In the case where $(\gamma,k)\in\mathbb{Z}_{<0}\times\mathbb{N}$, the Laplace transform $\mathcal{L}$ is defined differently [20, (4.13)], and does not satisfy the second formula of (3.4) below when $\gamma\in\mathbb{Z}_{<0}$ and $k=0$ [20, (4.11)]. Hence, the definition of $\mathcal{L}$ in the case where $(\gamma,k)\in\mathbb{Z}_{<0}\times\mathbb{N}$ will not be interesting in our context.

With this definition, we find formally [20, (4.12)]:

$$\mathcal{L}(h_{\gamma,k}) = \Gamma(\gamma+1)\, x^{-\gamma-1} \sum_{j=0}^{k} \rho_{\gamma,j}^{(k)}\, (\log x)^j \tag{3.3}$$
$$\text{with } \rho_{\gamma,k}^{(k)} = (-1)^k,\ \ \rho_{\gamma,j}^{(k)} \in <\Gamma(\gamma),\dots,\Gamma^{(k)}(\gamma)>_{\mathbb{Q}[\gamma]},\ \ j=0,\dots,k-1.$$

Moreover, this transformation $\mathcal{L}$ has the following formal properties (cf. [20, (4.10), (4.11)], [2, (5.3.7), (5.3.8)]):

$$\frac{d}{dx}(\mathcal{L}(h_{\gamma,k})) = \mathcal{L}(-xh_{\gamma,k}), \quad \text{and} \qquad \mathcal{L}\Big(\frac{d}{dx}(h_{\gamma,k})\Big) = x\mathcal{L}(h_{\gamma,k}). \tag{3.4}$$

Combining formulas (3.3) and (3.4), we obtain the following recurrence relation with respect to the index j,

$$\rho_{\gamma+1,j-1}^{(k)} = -\rho_{\gamma,j-1}^{(k)} + \frac{j}{\gamma+1}\, \rho_{\gamma,j}^{(k)}. \tag{3.5}$$

In the sequel, we fix γ in $K\setminus\mathbb{Z}$ and k in $\mathbb{N}$. With the notation above, the following Lemma shows that, for any $n\in\mathbb{N}$ and any $0\le j\le k$, the coefficient $\rho_{\gamma\pm n,j}^{(k)}$ can be written as a $\mathbb{Q}(\gamma)$-linear combination of $\rho_{\gamma,j}^{(k)},\dots,\rho_{\gamma,k}^{(k)}$.

Lemma 3.1. *For any $0\le j\le k$ and for any $n\in\mathbb{N}$, there exist numbers $r_{\gamma+n,j}^{(k,\ell)}$, $r_{\gamma-n,j}^{(k,\ell)}$ in $\mathbb{Q}(\gamma)$ for $\ell=j,\dots,k$, such that*

$$\rho_{\gamma+n,j}^{(k)} = \sum_{\ell=j}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma+n,j}^{(k,\ell)}, \quad \textit{and} \quad \rho_{\gamma-n,j}^{(k)} = \sum_{\ell=j}^{k} \rho_{\gamma,\ell}^{(k)}\, r_{\gamma-n,j}^{(k,\ell)}. \tag{3.6}$$

Moreover, for any finite place v in Σ_{f},

$$\limsup_{n\longrightarrow+\infty} \left| r_{\gamma\pm n,j}^{(k,\ell)} \right|_v^{1/n} \le 1. \tag{3.7}$$

The proof is similar to that of [20, Lemma 4.2]. We reproduce it here in order to clarify the nature of the coefficients $r_{\gamma\pm n,j}^{(k,\ell)}$. This will be useful in the proof of Proposition 3.8.

Proof. We prove the lemma by downward induction on the index j. In the case $j=k\ge0$, by (3.3), it suffices to take $r_{\gamma\pm n,k}^{(k,k)}=1$ for any $n\in\mathbb{N}$. Assume now that the lemma is true for some index j with $1\le j\le k$. From (3.5) and by iteration on $n\ge1$, we find

$$\begin{aligned}\rho_{\gamma+n,j-1}^{(k)} &= (-1)^n \rho_{\gamma,j-1}^{(k)} + j\sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\gamma+i+1}\, \rho_{\gamma+i,j}^{(k)}\\ &= (-1)^n \rho_{\gamma,j-1}^{(k)} + j\sum_{\ell=j}^{k} \rho_{\gamma,\ell}^{(k)} \sum_{i=0}^{n-1} \frac{(-1)^{n+i+1}}{\gamma+i+1}\, r_{\gamma+i,j}^{(k,\ell)},\end{aligned}$$

and

$$\rho^{(k)}_{\gamma-1,j-1} = -\rho^{(k)}_{\gamma,j-1} + \frac{j}{\gamma}\,\rho^{(k)}_{\gamma-1,j},$$

$$\begin{aligned}\rho^{(k)}_{\gamma-n,j-1} &= (-1)^n\rho^{(k)}_{\gamma,j-1} + j\sum_{i=1}^{n}\frac{(-1)^{n-i}}{\gamma-i+1}\rho^{(k)}_{\gamma-i,j}\\ &= (-1)^n\rho^{(k)}_{\gamma,j-1} + j\sum_{\ell=j}^{k}\rho^{(k)}_{\gamma,\ell}\sum_{i=1}^{n}\frac{(-1)^{n-i}}{\gamma-i+1}r^{(k,\ell)}_{\gamma-i,j}.\end{aligned}$$

Putting for all $j \leq \ell \leq k$ and all $n \geq 1$

$$\begin{aligned} r^{(k,j-1)}_{\gamma\pm n,j-1} &= (-1)^n\\ r^{(k,\ell)}_{\gamma+n,j-1} &= j\sum_{i=0}^{n-1}\frac{(-1)^{n+i+1}}{\gamma+i+1}\,r^{(k,\ell)}_{\gamma+i,j}\\ r^{(k,\ell)}_{\gamma-n,j-1} &= j\sum_{i=1}^{n}\frac{(-1)^{n-i}}{\gamma-i+1}\,r^{(k,\ell)}_{\gamma-i,j},\end{aligned}\tag{3.8}$$

we get, $r^{(k,\ell)}_{\gamma\pm n,j-1} \in \mathbb{Q}(\gamma)$ for $\ell = j-1,\dots,k$,

$$\rho^{(k)}_{\gamma+n,j-1} = \sum_{\ell=j-1}^{k}\rho^{(k)}_{\gamma,\ell}\,r^{(k,\ell)}_{\gamma+n,j-1} \quad\text{and}\quad \rho^{(k)}_{\gamma-n,j-1} = \sum_{\ell=j-1}^{k}\rho^{(k)}_{\gamma,\ell}\,r^{(k,\ell)}_{\gamma-n,j-1}.$$

This proves the first statement of the lemma. Let now $v \in \Sigma_{\mathrm{f}}$. By induction hypotheses, we have $\limsup_{n\longrightarrow\infty}\left|r^{(k,\ell)}_{\gamma\pm n,j}\right|_v^{1/n} \leq 1$ for $\ell = j,\dots,k$. Since γ is an element of K, it is non-Liouville for $p(v)$ and consequently we have $\limsup_{n\longrightarrow\infty}\left|\frac{1}{\gamma\pm n}\right|_v^{1/n} = 1$ (cf. [12, VI.1.1]). We deduce

$$\limsup_{n\longrightarrow\infty}\left(\max_{0\leq i\leq n}\left|r^{(k,\ell)}_{\gamma\pm i,j}\right|_v^{1/n}\right) \leq 1, \quad \ell = j,\dots,k$$

and

$$\limsup_{n\longrightarrow\infty}\left(\max_{0\leq i\leq n}\left|\frac{1}{\gamma\pm i+1}\right|_v^{1/n}\right) \leq 1.$$

Combining these estimations with (3.8) we get for $\ell = j,\dots,k$,

$$\limsup_{n\longrightarrow\infty}\left|r^{(k,\ell)}_{\gamma\pm n,j-1}\right|_v^{1/n} \leq 1.$$

The case $\ell = j-1$ is trivial by (3.8). This completes the proof of the lemma. $\square$

In addition, if we set $r^{(k,\ell)}_{\gamma\pm n,j}=0$ for $\ell=0,\dots,j-1$, we find, by combining formulae (3.3) and (3.6),

$$
\begin{aligned}
\mathcal{L}(h_{\gamma+n,k}) &= \Gamma(\gamma+n+1)\ x^{-\gamma-n-1}\ \sum_{j=0}^{k}\ \rho^{(k)}_{\gamma+n,j}\ (\log x)^j\\
&= (\gamma)_{n+1}\Gamma(\gamma)\ x^{-\gamma-n-1}\Big((-1)^k(\log x)^k\ +\ \sum_{j=0}^{k-1}\sum_{\ell=0}^{k}\rho^{(k)}_{\gamma,\ell}\ r^{(k,\ell)}_{\gamma+n,j}\ (\log x)^j\Big),\\
\mathcal{L}(h_{\gamma-n,k}) &= \Gamma(\gamma-n+1)\ x^{-\gamma+n-1}\ \sum_{j=0}^{k}\ \rho^{(k)}_{\gamma-n,j}\ \ (\log x)^j\\
&= \frac{(-1)^n\gamma\Gamma(\gamma)}{(-\gamma)_n}\ x^{-\gamma+n-1}\Big((-1)^k(\log x)^k\ +\ \sum_{j=0}^{k-1}\sum_{\ell=0}^{k}\rho^{(k)}_{\gamma,\ell}\ r^{(k,\ell)}_{\gamma-n,j}\ (\log x)^j\Big).
\end{aligned}
\tag{3.9}
$$

Remarks 3.2. (i) According to the recursion formulae (3.8) and by downward induction on j, we find, for all $0\le j,\ell\le k$ and all $n\in\mathbb{N}$

$$
\begin{aligned}
r^{(k,\ell)}_{\gamma+n,j} &\in \Big\langle 1,\ \prod_{1\le i\le t}\frac{1}{\gamma+m_i},\quad 1\le m_i\le n,\quad 1\le t\le k\Big\rangle_{\mathbb{Z}},\\
r^{(k,\ell)}_{\gamma-n,j} &\in \Big\langle 1,\ \prod_{1\le i\le t}\frac{1}{\gamma-m_i},\quad 0\le m_i\le n,\quad 1\le t\le k\Big\rangle_{\mathbb{Z}}.
\end{aligned}
\tag{3.10}
$$

(ii) The formula (3.3) shows that, for any $(\gamma,k)\in K\setminus\mathbb{Z}\times\mathbb{N}$, $\mathcal{L}(h_{\gamma,k})\neq 0$. Hence, by (3.4), we have $\mathcal{L}(xh_{\gamma,k})\frac{d}{dx}(\mathcal{L}(h_{\gamma,k}))\neq 0$.
(iii) From (3.9) and by x-adic formal completion, the Laplace transform $\mathcal{L}$ can be extended to injective K-linear maps:

$$
x^\gamma\ K[[x]]\hookrightarrow x^{-\gamma-1}\ \mathbb{C}[[\frac{1}{x}]],\quad x^\gamma\ K[[\frac{1}{x}]]\hookrightarrow x^{-\gamma-1}\ \mathbb{C}[[x]],
$$
$$
x^\gamma\ K[[x]][\log x]\hookrightarrow x^{-\gamma-1}\ \mathbb{C}[[\frac{1}{x}]][\log x],\quad x^\gamma\ K[[\frac{1}{x}]][\log x]\hookrightarrow x^{-\gamma-1}\ \mathbb{C}[[x]][\log x].
$$

The injectivity can be proved by filtering by the degree of the logarithms using (3.9).

Lemma 3.3. *With the notation of Lemma 3.1, we have*
(1) if $\gamma\in K\setminus\mathbb{Z}$, the absolute value (in the usual sense) of $r^{(k,\ell)}_{\gamma\ \pm n\ ,j}$ has at most a geometric growth in n for all $0\le j,\ell\le k$.
(2) if $\gamma\in\mathbb{Q}\setminus\mathbb{Z}$, the quantities $r^{(k,\ell)}_{\gamma+n,j}$ and $r^{(k,\ell)}_{\gamma-n,j}$ are rational numbers for any n, and the least common denominator of $r^{(k,\ell)}_{\gamma,j},r^{(k,\ell)}_{\gamma+1,j},\dots,r^{(k,\ell)}_{\gamma+n,j}$ (resp. $r^{(k,\ell)}_{\gamma,j},r^{(k,\ell)}_{\gamma-1,j},\dots,r^{(k,\ell)}_{\gamma-n,j}$) has at most a geometric growth in n for all $0\le j,\ell\le k$.

Proof. (1) follows by downward induction on j using (3.8) and the fact that

$$
\log\Big(\Big|\frac{1}{\gamma}\Big|+\Big|\frac{1}{\gamma+1}\Big|+\dots+\Big|\frac{1}{\gamma+n}\Big|\Big)=O(n)\quad\text{and}\quad\log\Big(\Big|\frac{1}{\gamma}\Big|+\Big|\frac{1}{\gamma-1}\Big|+\dots+\Big|\frac{1}{\gamma-n+1}\Big|\Big)=O(n).
$$

(2) Let $\gamma = a/b$ with $(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}$. It is clear by (3.10) that the quantities $r^{(k,\ell)}_{\gamma+n,j}$ and $r^{(k,\ell)}_{\gamma-n,j}$ are rational numbers for any n. In addition, the prime numbers Theorem (see e.g. [17]) implies that $\lim_{n\to+\infty} \frac{\log \operatorname{lcm}\{1,\ldots,n\}}{n} = 1$, where "lcm" denotes the least common multiple. Therefore, there exists a positive constant $C > 0$ such that for n sufficiently large, we have $\operatorname{lcm}\{1,\ldots,n\} \leq C^n$. Thus,

$$\operatorname{lcm}(|a|, |a-b|, |a-2b|, \ldots, |a+(1-n)b|) \leq C^{|a|}C^{bn},$$

and

$$\operatorname{lcm}(|a|, |a+b|, |a+2b|, \ldots, |a+nb|) \leq C^{|a|}C^{bn}.$$

Hence,

$$\operatorname{lcd}\Big\{ \prod_{1\leq i\leq t} \frac{1}{\gamma+m_i} \quad | \quad 1 \leq m_i \leq n, \quad 1 \leq t \leq k \Big\} \leq C^{k|a|}C^{kbn},$$

and

$$\operatorname{lcd}\Big\{ \prod_{1\leq i\leq t} \frac{1}{\gamma-m_i} \quad | \quad 0 \leq m_i \leq n, \quad 1 \leq t \leq k \Big\} \leq C^{k|a|}C^{kbn},$$

where "lcd" denotes the least common denominator. Combining this with (3.10), we get the last statement of (2). $\square$

3.2 Standard Laplace transform in several variables. In this paragraph, we extend the Laplace transform defined in the previous paragraph to several variables while preserving the K-linearity, and the commutation with derivations in the following sense:

(3.11)

$$\underline{\partial}^{\underline{\alpha}}(\underline{\mathcal{L}}(h_{\underline{\gamma},\underline{k}})) = \underline{\mathcal{L}}((-1)^{|\underline{\alpha}|}\underline{x}^{\underline{\alpha}}h_{\underline{\gamma},\underline{k}}), \quad \text{and} \quad \underline{\mathcal{L}}(\underline{\partial}^{\underline{\alpha}}(h_{\underline{\gamma},\underline{k}})) = \underline{x}^{\underline{\alpha}}\underline{\mathcal{L}}(h_{\underline{\gamma},\underline{k}}) \quad \text{for any} \quad \underline{\alpha} \in \mathbb{N}^d.$$

With these conditions, the definition of $\underline{\mathcal{L}}$ will be interesting just in the case where $(\underline{\gamma}, \underline{k}) \in (K \setminus \mathbb{Z})^d \times \mathbb{N}^d$. Indeed, if for some $1 \leq i_0 \leq d$, $(\gamma_{i_0}, k_{i_0}) \in \mathbb{N} \times \{0\}$, we will find

$$x_{i_0}^{\gamma_{i_0}+1}\underline{\mathcal{L}}(h_{\underline{\gamma},\underline{k}}) = \underline{\mathcal{L}}(\partial_{i_0}^{\gamma_{i_0}+1}(h_{\underline{\gamma},\underline{k}})) = \underline{\mathcal{L}}(0) = 0.$$

For the remainder of this section, we fix $(\underline{\gamma}, \underline{k})$ in $(K \setminus \mathbb{Z})^d \times \mathbb{N}^d$. For

$$f(\underline{x}) = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{-\underline{\alpha}}\underline{x}^{-\underline{\alpha}} + \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}}\underline{x}^{\underline{\alpha}} \in K[[\underline{x}, \frac{1}{\underline{x}}]],^{2}$$

we define the Laplace transform $\underline{\mathcal{L}}(f(\underline{x})h_{\underline{\gamma},\underline{k}})$ of $f(\underline{x})h_{\underline{\gamma},\underline{k}}$ as follows:

$$\underline{\mathcal{L}}(f(\underline{x})h_{\underline{\gamma},\underline{k}}) = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{-\underline{\alpha}} \prod_{1\leq i\leq d} \mathcal{L}_i(h_{\gamma_i-\alpha_i,k_i}) + \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}} \prod_{1\leq i\leq d} \mathcal{L}_i(h_{\gamma_i+\alpha_i,k_i}),$$

where $\mathcal{L}_i$ denotes the Laplace transform defined in the previous paragraph with respect to the variable x_i. It is easy to check, from (3.4) and (2) of Lemma 3.3, that $\underline{\mathcal{L}}$ satisfies the formulae of (3.11).

[2]Note here that the Cauchy multiplication is not involved in this theory.

Explicitly, if we set $\Gamma(\underline{\gamma}) = \Gamma(\gamma_1)\dots\Gamma(\gamma_d)$, we get from (3.9):

$$\begin{aligned}
\underline{\mathcal{L}}(h_{\underline{\gamma}+\underline{\alpha},\underline{k}}) =&(\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}} \prod_{1\le i\le d}\Big(\sum_{j_i=0}^{k_i} \rho^{(k_i)}_{\gamma_i+\alpha_i,j_i}\ \ (\log x_i)^{j_i}\Big),\\
&= (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}}\Big((-1)^{|\underline{k}|}\ (\log \underline{x})^{\underline{k}} + \sum_{\underline{j}<\underline{k}}\Big(\prod_{1\le i\le d}\rho^{(k_i)}_{\gamma_i+\alpha_i,j_i}\Big)\ (\log\underline{x})^{\underline{j}}\Big)\\
&= (-1)^{|\underline{k}|}(\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ h_{-\underline{\gamma}-\underline{\alpha}-\underline{1},\underline{k}}\\
&\quad + \ (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}}\sum_{\underline{j}<\underline{k}}\Big(\sum_{\underline{\ell}\le\underline{k}}\prod_{1\le i\le d}\rho^{(k_i)}_{\gamma_i,\ell_i}\ r^{(k_i,\ell_i)}_{\gamma_i+\alpha_i,j_i}\Big)\ (\log\underline{x})^{\underline{j}}\\
\underline{\mathcal{L}}(h_{\underline{\gamma}-\underline{\alpha},\underline{k}}) =&\frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\prod_{1\le i\le d}\Big(\sum_{j_i=0}^{k_i}\rho^{(k_i)}_{\gamma_i-\alpha_i,j_i}\ \ (\log x_i)^{j_i}\Big)\\
&= \frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\Big((-1)^{|\underline{k}|}\ (\log\underline{x})^{\underline{k}} + \sum_{\underline{j}<\underline{k}}\Big(\prod_{1\le i\le d}\rho^{(k_i)}_{\gamma_i-\alpha_i,j_i}\Big)\ (\log\underline{x})^{\underline{j}}\Big)\\
&= \ \frac{(-1)^{|\underline{\alpha}|+|\underline{k}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ h_{-\underline{\gamma}+\underline{\alpha}-\underline{1},\underline{k}}\\
&\quad + \ \frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ \ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\sum_{\underline{j}<\underline{k}}\Big(\sum_{\underline{\ell}\le\underline{k}}\prod_{1\le i\le d}\rho^{(k_i)}_{\gamma_i,\ell_i}\ r^{(k_i,\ell_i)}_{\gamma_i-\alpha_i,j_i}\Big)\ (\log\underline{x})^{\underline{j}}.
\end{aligned}$$

If we set $\rho^{(\underline{k})}_{\underline{\gamma},\underline{\ell}} = \prod_{1\le i\le d}\rho^{(k_i)}_{\gamma_i,\ell_i}$, $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}-\underline{\alpha},\underline{j}} = \prod_{1\le i\le d} r^{(k_i,\ell_i)}_{\gamma_i-\alpha_i,j_i}$ and $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}} = \prod_{1\le i\le d} r^{(k_i,\ell_i)}_{\gamma_i+\alpha_i,j_i}$, the equalities above become

$$\begin{aligned}
\underline{\mathcal{L}}(h_{\underline{\gamma}+\underline{\alpha},\underline{k}}) &= (-1)^{|\underline{k}|}(\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ h_{-\underline{\gamma}-\underline{\alpha}-\underline{1},\underline{k}}\\
&\quad + \ (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\Gamma(\underline{\gamma})\ \underline{x}^{-\underline{\gamma}-\underline{\alpha}-\underline{1}}\sum_{\underline{j}<\underline{k}}\Big(\sum_{\underline{\ell}\le\underline{k}}\rho^{(\underline{k})}_{\underline{\gamma},\underline{\ell}}\ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}}\Big)\ (\log\underline{x})^{\underline{j}}\\
\underline{\mathcal{L}}(h_{\underline{\gamma}-\underline{\alpha},\underline{k}}) &= \ \frac{(-1)^{|\underline{\alpha}|+|\underline{k}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ h_{-\underline{\gamma}+\underline{\alpha}-\underline{1},\underline{k}}\\
&\quad + \ \frac{(-1)^{|\underline{\alpha}|}\Gamma(\underline{\gamma})(\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ \ \underline{x}^{-\underline{\gamma}+\underline{\alpha}-\underline{1}}\sum_{\underline{j}<\underline{k}}\Big(\sum_{\underline{\ell}\le\underline{k}}\rho^{(\underline{k})}_{\underline{\gamma},\underline{\ell}}\ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}-\underline{\alpha},\underline{j}}\Big)\ (\log\underline{x})^{\underline{j}}.
\end{aligned}\tag{3.12}$$

Remark 3.4. From (3.12) and by $(x_1,\dots,x_d)$-adic formal completion, the Laplace transform $\underline{\mathcal{L}}$ extends to injective K-linear maps:

$$\underline{x}^{\underline{\gamma}}\ K[[\underline{x}]] \hookrightarrow \underline{x}^{-\underline{\gamma}-\underline{1}}\ \mathbb{C}[[\frac{1}{\underline{x}}]],\quad \underline{x}^{\underline{\gamma}}\ K[[\frac{1}{\underline{x}}]] \hookrightarrow \underline{x}^{-\underline{\gamma}-\underline{1}}\ \mathbb{C}[[\underline{x}]],$$

$$\underline{x}^{\underline{\gamma}}\ K[[\underline{x}]][\log x_1,\dots,\log x_d] \hookrightarrow \underline{x}^{-\underline{\gamma}-\underline{1}}\ \mathbb{C}[[\frac{1}{\underline{x}}]][\log x_1,\dots,\log x_d],$$

$$\underline{x}^{\underline{\gamma}}\ K[[\frac{1}{\underline{x}}]][\log x_1,\dots,\log x_d] \hookrightarrow \underline{x}^{-\underline{\gamma}-\underline{1}}\ \mathbb{C}[[\underline{x}]][\log x_1,\dots,\log x_d].$$

The injectivity follows by filtering by the order (defined in section 2) of the multi-exponent of $\log \underline{x}$ using (3.12).

3.3 Arithmetic estimates. In this paragraph, for $v \in \Sigma_{\mathrm{f}}$ and $f \in K[[\underline{x}]]$, we give, in term of $r_v(f)$, lower bounds of $p(v)$-adic radius of convergence of the formal power series over K occurring in $\underline{\mathcal{L}}(fh_{\underline{\gamma},\underline{k}})$ and $\underline{\mathcal{L}}\Big(f\Big(\frac{1}{\underline{x}}\Big)h_{\underline{\gamma},\underline{k}}\Big)$.

Lemma 3.5. *With the notation of the previous paragraph (in particular formula (3.12)), we have*
1) For any $v \in \Sigma_{\mathrm{f}}$,

$$\limsup_{|\underline{\alpha}|\to+\infty} \left| r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}} \right|_v^{1/|\underline{\alpha}|} \leq 1 \quad \textit{and} \quad \limsup_{|\underline{\alpha}|\to+\infty} \left| r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}-\underline{\alpha},\underline{j}} \right|_v^{1/|\underline{\alpha}|} \leq 1.$$

2) The absolute value (in the usual sense) of $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}}$ has at most a geometric growth in $|\underline{\alpha}|$.
3) If $\underline{\gamma} \in (\mathbb{Q}\setminus\mathbb{Z})^d$, the quantities $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}\pm\underline{\alpha},\underline{j}}$ are rational numbers for all $\underline{\alpha} \in \mathbb{N}^d$, and the least common denominator of $\left\{ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}} \;\middle|\; |\underline{\alpha}| \leq n \right\}$ $\left(\text{resp. } \left\{ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}-\underline{\alpha},\underline{j}} \;\middle|\; |\underline{\alpha}| \leq n \right\}\right)$ has at most a geometric growth in n.

Proof. 1) By Lemma 3.1, we have, for $i = 1, \ldots, d$, $\limsup_{\alpha_i\to+\infty} \left| r^{(k_i,\ell_i)}_{\gamma_i\pm\alpha_i,j_i} \right|_v^{1/|\alpha_i|} \leq 1$, and hence $\limsup_{|\underline{\alpha}|\to+\infty} \left| r^{(k_i,\ell_i)}_{\gamma_i\pm\alpha_i,j_i} \right|_v^{1/|\underline{\alpha}|} \leq 1$. The product over all the $1 \leq i \leq d$ of these quantities gives the desired inequalities. The rest of the lemma results from Lemma 3.3 and the definitions above of the quantities $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}\pm\underline{\alpha},\underline{j}}$. □

The arithmetic properties of $\underline{\mathcal{L}}$ are based on the following Lemma which generalizes a well known identity in the p-adic analysis.

Lemma 3.6. *Let $v \in \Sigma_{\mathrm{f}}$. Assume that $\gamma_1, \ldots, \gamma_d$ are (non-Liouville) numbers of $K \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$. Then*

$$\lim_{|\underline{\alpha}|\to+\infty} \left| (\underline{\gamma})_{\underline{\alpha}+\underline{1}} \right|_v^{1/|\underline{\alpha}|} = \pi_v.$$

In particular, for $d = 1$, we have $\lim_{\alpha_1\to+\infty} |(\gamma_1)_{\alpha_1}|_v^{1/\alpha_1} = \pi_v$.

Proof. Combining the inequalities (12) and (14) of [10], we find that, for any $i = 1, \ldots, d$, there exist two real numbers e_i, e_i' such that

$$|p(v)|_v^{(\alpha_i/(p(v)-1)+e_i\log(1+\alpha_i)+e_i')} \leq |\gamma_i^{-1}(\gamma_i)_{(\alpha_i+1)}|_v \leq \frac{|p(v)|_v^{\alpha_i/(p(v)-1)}}{(\alpha_i+1)}, \quad \text{for any} \quad \alpha_i \in \mathbb{N}.$$

Thus,

$$|p(v)|_v^{(|\underline{\alpha}|/(p(v)-1)+\sum_{1\leq i\leq d}(e_i\log(1+\alpha_i)+e_i'))} \Big| \prod_{1\leq i\leq d} \gamma_i \Big|_v \leq |(\underline{\gamma})_{\underline{\alpha}+\underline{1}}|_v \leq \prod_{1\leq i\leq d} \frac{|\gamma_i|_v}{(\alpha_i+1)} |p(v)|_v^{|\underline{\alpha}|/(p(v)-1)},$$

for any $\underline{\alpha} \in \mathbb{N}^d$. Hence, the lemma results from the fact:

$$\Big| \sum_{1\le i\le d} e_i \log(1+\alpha_i)\Big| \le d \max_{1\le i\le d}(1,|e_i|)\log(1+|\underline{\alpha}|).$$

□

Before stating the following proposition, let us recall that the notations $\mathcal{R}_v^s(f)$, $r_v(f)$ and $\mathcal{F}$ are defined in sections 1 and 2.

Proposition 3.7. *Let v be a finite place of Σ_{f} and let $f = \sum_{\underline{\alpha}\in\mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K[[\underline{x}]]$ be a power series. Assume $\underline{\gamma} \in (K \cap \ \mathbb{Z}_{p(v)} \setminus \mathbb{Z})^d$. Then there exist power series $f_{\underline{\gamma},\underline{k},\underline{j}} \in \mathbb{C}\otimes_K \mathcal{R}_v^{-1}(f)$ (resp. $f^*_{\underline{\gamma},\underline{k},\underline{j}} \in \mathbb{C}\otimes_K \mathcal{R}_v^{1}(f)$), $\underline{j}\le\underline{k}$, which satisfy the following conditions*

$$\begin{aligned}
\underline{\mathcal{L}}\Big(f h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}}\underline{\Gamma}(\underline{\gamma}) \sum_{\underline{j}\le\underline{k}} f_{\underline{\gamma},\underline{k},\underline{j}}\Big(\frac{1}{\underline{x}}\Big)(\log\underline{x})^{\underline{j}} \\
\underline{\mathcal{L}}\Big(f\Big(\frac{1}{\underline{x}}\Big)h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}}\underline{\Gamma}(\underline{\gamma}) \sum_{\underline{j}\le\underline{k}} f^*_{\underline{\gamma},\underline{k},\underline{j}}\ (\log\underline{x})^{\underline{j}}
\end{aligned} \tag{3.13}$$

with $f_{\underline{\gamma},\underline{k},\underline{k}} f^*_{\underline{\gamma},\underline{k},\underline{k}} \neq 0$ such that $r_v(f_{\underline{\gamma},\underline{k},\underline{k}}) = r_v(f)\pi_v^{-1}$ (resp. $r_v(f^*_{\underline{\gamma},\underline{k},\underline{k}}) = r_v(f)\pi_v$). Moreover, if $f h_{\underline{\gamma},\underline{k}}$ (resp. $f(\frac{1}{\underline{x}})h_{\underline{\gamma},\underline{k}}$) is solution of a differential equation ϕ of $K[x_1,\ldots,x_d,\partial_1,\ldots,\partial_d]$, then $\mathcal{L}(fh_{\underline{\gamma},\underline{k}})$ (resp. $\mathcal{L}(f(\frac{1}{\underline{x}})h_{\underline{\gamma},\underline{k}})$) is solution of $\mathcal{F}(\phi)$, where $\mathcal{F}$ is the Fourier-Laplace transform.

Proof. Set, for all $|\underline{j}|\le|\underline{k}|$,

$$\begin{aligned}
f_{\underline{\gamma},\underline{k},\underline{k}} &= \sum_{\underline{\alpha}\in\mathbb{N}^d} (-1)^{|\underline{k}|}\, a_{\underline{\alpha}}\ (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\ \underline{x}^{\underline{\alpha}} \\
f_{\underline{\gamma},\underline{k},\underline{j}} &= \sum_{\underline{\ell}\le\underline{k}} \rho^{(\underline{k})}_{\underline{\gamma},\underline{\ell}} \sum_{\underline{\alpha}\in\mathbb{N}^d} (-1)^{|\underline{k}|}\, a_{\underline{\alpha}}\ (\underline{\gamma})_{\underline{\alpha}+\underline{1}}\ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}+\underline{\alpha},\underline{j}}\ \underline{x}^{\underline{\alpha}} \\
f^*_{\underline{\gamma},\underline{k},\underline{k}} &= \sum_{\underline{\alpha}\in\mathbb{N}^d} \frac{(-1)^{|\underline{\alpha}|+|\underline{k}|}\ (\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ a_{\underline{\alpha}}\ \underline{x}^{\underline{\alpha}} \\
f^*_{\underline{\gamma},\underline{k},\underline{j}} &= \sum_{\underline{\ell}\le\underline{k}} \rho^{(\underline{k})}_{\underline{\gamma},\underline{\ell}} \sum_{\underline{\alpha}\in\mathbb{N}^d} \frac{(-1)^{|\underline{\alpha}|+|\underline{k}|}\ (\underline{\gamma})_{\underline{0}}}{(-\underline{\gamma})_{\underline{\alpha}}}\ a_{\underline{\alpha}}\ r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}-\underline{\alpha},\underline{j}}\ \underline{x}^{\underline{\alpha}}.
\end{aligned} \tag{3.14}$$

By (3.12), and in virtue of Lemmas 3.5 and 3.6, one sees that these power series satisfy the desired conditions. The last statement results from (3.11). □

3.4 Laplace transform and arithmetic Gevrey series. In this paragraph, we shall explain the action of the standard Laplace transform in several variables on the arithmetic Gevrey series. Let us first give the definition of these power series:

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and let $(a_{\underline{\alpha}})_{\underline{\alpha}\in\mathbb{N}^d}$ be a family of algebraic numbers of $\overline{\mathbb{Q}}$. Consider the following conditions:

(A_1): for all $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, there exists a constant $C_1 \in \mathbb{R}_{>0}$ such that $a_{\underline{\alpha}}$ and its conjugates over $\mathbb{Q}$ do not exceed $C_1^{|\underline{\alpha}|}$ in absolute value;

(A_2): there exists a constant $C_2 \in \mathbb{R}_{>0}$ such that the common denominator in $\mathbb{N}$ of $\{a_{\underline{\alpha}}, |\underline{\alpha}| \leq n\}$ does not exceed C_2^{n+1}.

Definition. An *arithmetic Gevrey series of order* $s \in \mathbb{Q}$ is an element $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$ of $K[[\underline{x}]]$ such that the sequence $(a_{\underline{\alpha}}/(\underline{\alpha}!)^s)_{\underline{\alpha} \in \mathbb{N}^d}$ satisfies the conditions (A_1) and (A_2).

The power series $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$ is called a *G-function* (resp. *E-function*) if it is an arithmetic Gevrey series of order 0 (resp. -1) and satisfies the following holonomicity condition:

(H): f is called rationally holonomic over $K(\underline{x})$, if the $K(\underline{x})/K$-differential module generated by $(\underline{\partial}^{\underline{\alpha}}(f))_{\underline{\alpha} \in \mathbb{N}^d}$ is a $K(\underline{x})$-vector space of finite dimension.

The condition (H) is equivalent to say that f is a solution of a linear partial differential equation with coefficients in $K(\underline{x})$.

Notation. Let s be a rational number. We denote $K\{\underline{x}\}_s$ the set of the power series $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K[[\underline{x}]]$ such that the sequence $(a_{\underline{\alpha}}/(\underline{\alpha}!)^s)_{\underline{\alpha} \in \mathbb{N}^d}$ satisfies (A_1) and (A_2).

$$K\Big\{\frac{1}{\underline{x}}\Big\}_s = \Big\{ \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{-\underline{\alpha}} \mid \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K\{\underline{x}\}_s \Big\},$$

$$NGA\{\underline{x}\}_s^K = \Big\{ \sum_{\text{finite sum}} f_{\underline{\gamma},\underline{k}} h_{\underline{\gamma},\underline{k}} \;\Big|\; f_{\underline{\gamma},\underline{k}} \in K\{\underline{x}\}_s, \ \ (\underline{\gamma}, \underline{k}) \in (\mathbb{Q} \setminus \mathbb{Z})^d \times \mathbb{N}^d \Big\},$$

$$NGA\Big\{\frac{1}{\underline{x}}\Big\}_s^K = \Big\{ \sum_{\text{finite sum}} f_{\underline{\gamma},\underline{k}} h_{\underline{\gamma},\underline{k}} \;\Big|\; f_{\underline{\gamma},\underline{k}} \in K\Big\{\frac{1}{\underline{x}}\Big\}_s, \ \ (\underline{\gamma}, \underline{k}) \in (\mathbb{Q} \setminus \mathbb{Z})^d \times \mathbb{N}^d \Big\}.$$

Following [2, section 1], we see that these two latter sets are differential $K[\underline{x}]$-algebras. The elements of $NGA\{\underline{x}\}_s^K$ will be called *Nilson-Gevrey power series of order* s. The following result expresses how the Laplace transform $\mathcal{L}$ acts on these differential $K[\underline{x}]$-algebras.

Proposition 3.8. *The Laplace transform* $\underline{\mathcal{L}}$ *induces the following injective* K*-linear maps*

$$NGA\{\underline{x}\}_s^K \hookrightarrow \mathbb{C} \otimes_K NGA\Big\{\frac{1}{\underline{x}}\Big\}_{s+1}^K,$$

$$NGA\Big\{\frac{1}{\underline{x}}\Big\}_s^K \hookrightarrow \mathbb{C} \otimes_K NGA\{\underline{x}\}_{s-1}^K.$$

Proof. Let $f = \sum_{\underline{\alpha} \in \mathbb{N}^d} a_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in K\{\underline{x}\}_s$ and $(\underline{\gamma}, \underline{k}) \in (\mathbb{Q} \setminus \mathbb{Z})^d \times \mathbb{N}^d$. By Proposition 3.7, we have

$$\begin{aligned}
\underline{\mathcal{L}}\Big(f h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}} \underline{\Gamma}(\underline{\gamma}) \sum_{\underline{j} \leq \underline{k}} f_{\underline{\gamma},\underline{k},\underline{j}}\Big(\frac{1}{\underline{x}}\Big) (\log \underline{x})^{\underline{j}} \\
\underline{\mathcal{L}}\Big(f\Big(\frac{1}{\underline{x}}\Big) h_{\underline{\gamma},\underline{k}}\Big) &= \underline{x}^{-\underline{\gamma}-\underline{1}} \underline{\Gamma}(\underline{\gamma}) \sum_{\underline{j} \leq \underline{k}} f^*_{\underline{\gamma},\underline{k},\underline{j}} \, (\log \underline{x})^{\underline{j}}
\end{aligned} \tag{3.15}$$

where the power series $f_{\underline{\gamma},\underline{k},\underline{j}}$ and $f^*_{\underline{\gamma},\underline{k},\underline{j}}$ ($\underline{j} \leq \underline{k}$) are defined by (3.14). Since the quantities $(\underline{\gamma})_{\underline{\alpha}}$ and $r^{(\underline{k},\underline{\ell})}_{\underline{\gamma}\,\pm\,\underline{\alpha},\underline{j}}$ are rational numbers for all $\underline{\alpha} \in \mathbb{N}^d$, and all $\underline{j}, \underline{\ell} \leq \underline{k}$, then we deduce from 2) and 3) of Lemma 3.5 and Lemma 3.9 bellow, that all the power series $f_{\underline{\gamma},\underline{k},\underline{j}}$ (resp. $f^*_{\underline{\gamma},\underline{k},\underline{j}}$) are in $\mathbb{C} \otimes_K NGA\left\{\frac{1}{\underline{x}}\right\}^K_{s+1}$ (resp. $\mathbb{C} \otimes_K NGA\{\underline{x}\}^K_{s-1}$). This implies, by the K-linearity of $\underline{\mathcal{L}}$, $\underline{\mathcal{L}}(NGA\{\underline{x}\}^K_s) \subseteq \mathbb{C} \otimes_K NGA\left\{\frac{1}{\underline{x}}\right\}^K_{s+1}$ and $\underline{\mathcal{L}}\Big(NGA\left\{\frac{1}{\underline{x}}\right\}^K_s\Big) \subseteq \mathbb{C} \otimes_K NGA\{\underline{x}\}^K_{s-1}$. The injectivity follows from Remark 3.4. $\square$

Lemma 3.9. [23] *Let $a_1, \ldots, a_d, b_1, \ldots, b_{d'}$ be rational numbers in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then there exists a positive constant $C > 0$ such that for any positive integer n,*

$$\operatorname{lcd}\Big(\frac{\prod_i (a_i)_0}{\prod_j (b_j)_0}, \ldots, \frac{\prod_i (a_i)_n}{\prod_j (b_j)_n}\Big) < n!^{d'-d} C^n,$$

where "lcd" denotes the least common denominator.

Question 3.10. Does the Laplace transform preserve the holonomicity condition (H) above?

4. Formal Laplace transform

4.1. Review of the one-variable case. In this paragraph, we recall the formal Laplace transform in one variable as it was defined in [21, §5]. This formal transformation allows to avoid the transcendental coefficients as those arising in the Laplace transform seen in the previous section. Moreover, this transformation has properties of commuting to derivation and therefore sends a basis of logarithmic solutions at 0 (resp. at infinity) of a differential equation $\phi \in K[x, d/dx]$ to logarithmic solutions at infinity (resp. at 0) of $\mathcal{F}_\tau(\phi)$ (for any $\tau \in K \setminus \{0\}$).

Let ν be a positive integer, τ an element of $K \setminus \{0\}$ and Λ an $\nu \times \nu$ matrix with entries in K such that all its eigenvalues belong to $K \setminus \mathbb{Z}$. Then the matrix

$$x^\Lambda = \exp(\Lambda \log x) = \sum_{n \geq 0} \frac{\Lambda^n (\log x)^n}{n!} \in \mathrm{GL}_\nu(K[[\log x]])$$

which satisfies

$$\frac{d}{dx}(x^\Lambda) = \Lambda x^{-1} x^\Lambda = \Lambda x^{\Lambda - \mathbb{I}_\nu}.$$

For any integer $\mu \geq 1$ and any $\mu \times \nu$ matrix $Y(x) = \sum_{n=-\infty}^{\infty} Y_i x^n$ with entries in $K[[x, 1/x]]$, the *Laplace transform $\mathcal{L}^\tau_\Lambda(Y(x)x^\Lambda)$ of $f := Y(x)x^\Lambda$*, with respect to Λ and to τ, is defined by

$$\mathcal{L}^\tau_\Lambda(Y(x)x^\Lambda) = \mathcal{L}^\tau_\Lambda\Big(\sum_{n=-\infty}^{\infty} Y_n x^{\Lambda + n\mathbb{I}_\nu}\Big) := \sum_{n=-\infty}^{\infty} Y_n C_{\Lambda,\tau}(n) x^{-\Lambda-(n+1)\mathbb{I}_\nu} \tag{4.1}$$

where $C_{\Lambda,\tau}\colon \mathbb{Z} \to \mathrm{GL}_n(K)$ is defined by the following conditions

$$C_{\Lambda,\tau}(n) = \begin{cases} \tau^{-n}(\Lambda + n\mathbb{I}_\nu)(\Lambda + (n-1)\mathbb{I}_\nu)\cdots(\Lambda + \mathbb{I}_\nu) & \text{si } n \geq 1, \\ \mathbb{I}_\nu & \text{si } n = 0, \\ \tau^{-n}\Lambda^{-1}(\Lambda - \mathbb{I}_\nu)^{-1}\cdots(\Lambda + (n+1)\mathbb{I}_\nu)^{-1} & \text{si } n \leq -1. \end{cases}$$

Notice that $C_{\Lambda,\tau}$, with these properties, satisfies

$$C_{\Lambda,\tau}(n)C_{-\Lambda,\tau}(-n-1) = (-1)^{n+1}\tau\Lambda^{-1}. \tag{4.2}$$

The transformation $\mathcal{L}^\tau_\Lambda$ has the following formal properties [21, 5.1.1]:
(4.3)

$$\mathcal{L}^\tau_{-\Lambda}\big(\mathcal{L}^\tau_\Lambda(f)\big) = -\tau Y(-x)\Lambda^{-1}x^\Lambda, \quad \mathcal{L}^\tau_\Lambda\Big(\frac{df}{dx}\Big) = \tau x\mathcal{L}^\tau_\Lambda(f) \text{ and } \mathcal{L}^\tau_\Lambda(xf) = -\frac{1}{\tau}\frac{d}{dx}\mathcal{L}^\tau_\Lambda(f).$$

Moreover, if $\mu = 1$ and if the entries of f are solutions of a differential equation $\phi \in K[x, d/dx]$, then those of $\mathcal{L}^\tau_\Lambda(f)$ are solutions of $\mathcal{F}_\tau(\phi)$.

4.2 Arithmetic estimates. For the remainder of this section, v denotes a fixed finite place of Σ_{f}. For any matrix M with entries in K, we denote by $\|M\|_v$ the maximum of the v-adic absolute values of the entries of M. Let $\Lambda \in \mathrm{GL}_\nu(\mathbb{Q})$ be an invertible matrix such that all its eigenvalues lie in $\mathbb{Q} \cap (\mathbb{Z}_{p(v)} \setminus \mathbb{Z})$. In this paragraph, we give upper and lower bounds of $C_{\Lambda,\tau}(n)$ with respect to the norm $\|.\|_v$ for any $n \in \mathbb{Z}$.

Lemma 4.1. *Let $\Lambda \in \mathrm{GL}_\nu(\mathbb{Q})$ be an invertible matrix such that all its eigenvalues $\gamma_1, \ldots, \gamma_s$ lie in $\mathbb{Q} \cap (\mathbb{Z}_{p(v)} \setminus \mathbb{Z})$ $(s \leq \nu)$. Then, there exist two positive real numbers c_1, c_2 such that for any $n \geq 1$*

$$c_1|\tau|_v^{-n} \max_{1\leq j\leq s}\{|(\gamma_j+1)_n|_v\} \leq \|C_{\Lambda,\tau}(n)\|_v \leq c_2 n^{\nu-1}|\tau|_v^{-n} \max_{1\leq j\leq s}\{|(\gamma_j+1)_n|_v\}. \tag{4.4}$$

In particular,

$$\lim_{n\to+\infty} \|C_{\Lambda,\tau}(n)\|_v^{1/n} = |\tau|_v^{-1}\pi_v.$$

Proof. Since the eigenvalues of Λ are all rational numbers, there exists $U \in \mathrm{GL}_n(\mathbb{Q})$ such that the product $\Delta = U^{-1}\Lambda U$ is in Jordan form. Setting $C_{\Delta,\tau} = U^{-1}C_{\Lambda,\tau}U$, there exist two positive real numbers c_0, c_1, such that $c_1 \leq \|C_{\Lambda,\tau}(n)\|_v/\|C_{\Delta,\tau}(n)\|_v \leq c_0$ for all $n \in \mathbb{Z}$. In addition, the matrix Δ is a block diagonal matrix with blocks $J_1 = \gamma_1\mathbb{I}_{\nu_1} + N_1, \ldots, J_s = \gamma_s\mathbb{I}_{\nu_s} + N_s$ on the diagonal (with $\nu_1 + \ldots + \nu_s = \nu$ and $N_1, \ldots, N_s$ are nilpotent matrices). Hence, for any $n \in \mathbb{Z}$, $C_{\Delta,\tau}(n)$ is a block diagonal matrix with blocks $C_{J_1,\tau}(n), \ldots, C_{J_s,\tau}(n)$ on the diagonal and $\|C_{\Delta,\tau}(n)\|_v = \max_{1\leq j\leq s}\|C_{J_j,\tau}(n)\|_v$. One the other hand, for $j = 1, \ldots, s$,

we have $(N_j)^\nu = 0$ and for any $n \geq 1$,

$$\begin{aligned} C_{J_j,\tau}(n) &= \tau^{-n}\prod_{\ell=1}^{n}((\gamma_j+\ell)\mathbb{I}_{\nu_j}+N_j) \\ &= \tau^{-n}(\gamma_j+1)_n\Big(\mathbb{I}_{\nu_j}+\sum_{t=1}^{\nu_j-1}\Big(\sum_{1\leq\ell_1<\cdots<\ell_t\leq n}\frac{1}{(\gamma_j+\ell_1)\cdots(\gamma_j+\ell_t)}\Big)N_j^t\Big). \end{aligned} \tag{4.5}$$

Now, for any integer $\ell \geq 1$, the sum $\gamma_j+\ell$ is a rational number for which the absolute value of numerator (in the usual sense) is bounded above by $\theta_j\ell$, for a constant $\theta_j > 0$ which only depends on γ_j, and for which the denominator is prime to $p(v)$. We deduce $|\gamma_j+\ell|_v \geq (\theta_j\ell)^{-1}$ for any $\ell \geq 1$ and hence, the decomposition (4.5) implies, for any $n \geq 1$,

$$|\tau^{-n}(\gamma_j+1)_n|_v \leq \|C_{J_j,\tau}(n)\|_v \leq (\theta_j n)^{\nu_j-1}|\tau^{-n}(\gamma_j+1)_n|_v. \tag{4.6}$$

The left inequality results from the fact that all the elements of the diagonal of $C_{J_j,\tau}(n)$ are equals to $\tau^{-n}(\gamma_j+1)_n$. This last observation gives $\det C_{J_j,\tau}(n) = \tau^{-n\nu_j}\Big((\gamma_j+1)_n\Big)^{\nu_j}$ and $\det C_{\Lambda,\tau}(n) = \det C_{\Delta,\tau}(n) = \tau^{-n\nu}\prod_{1\leq j\leq s}\Big((\gamma_j+1)_n\Big)^{\nu_j}$. In addition, the inequalities of (4.6) imply,

$$|\tau|_v^{-n}\max_{1\leq j\leq s}\{|(\gamma_j+1)_n|_v\} \leq \|C_{\Delta,\tau}(n)\|_v \leq n^{\nu-1}|\tau|_v^{-n}\max_{1\leq j\leq s}\{\theta_j^{\nu_j-1}|(\gamma_j+1)_n|_v\},$$

and therefore,

$$c_1|\tau|_v^{-n}\max_{1\leq j\leq s}\{|(\gamma_j+1)_n|_v\} \leq \|C_{\Lambda,\tau}(n)\|_v \leq c_0 n^{\nu-1}|\tau|_v^{-n}\max_{1\leq j\leq s}\{\theta_j^{\nu_j-1}\}\max_{1\leq j\leq s}\{|(\gamma_j+1)_n|_v\}. \tag{4.7}$$

Hence, putting $c_2 = c_0\max_{1\leq j\leq s}\{\theta_j^{\nu_j-1}\}$, we get (4.4), and therefore, by Lemma 3.6, the last statement of Lemma 4.1. □

Remark 4.2. *Notice that, for any matrix $Y \in \mathrm{GL}_\nu(K)$, we have*

$$\|Y\|_v^{-1} \leq \|Y^{-1}\|_v \leq |\det Y|_v^{-1}\|Y\|_v^{\nu-1}. \tag{4.8}$$

Indeed, the relation $\mathbb{I}_\nu = YY^{-1}$ implies $1 = \|\mathbb{I}_\nu\|_v \leq \|Y\|_v\|Y^{-1}\|_v$ which gives the left inequality above. The right inequality comes from the formula $Y^{-1} = (\det Y)^{-1}\mathrm{Adj}(Y)$ where $\mathrm{Adj}(Y)$ denotes the adjoint of Y. Now, by (4.8), if $Z \in \mathrm{GL}_\nu(K)$, we have

$$\|Y\|_v|\det Z|_v\|Z\|_v^{1-\nu} \leq \|Y\|_v\|Z^{-1}\|_v^{-1} \leq \|YZ\|_v \leq \|Y\|_v\|Z\|_v. \tag{4.9}$$

Lemma 4.3. *Let $\Lambda \in \mathrm{GL}_\nu(\mathbb{Q})$ be an invertible matrix such that all its eigenvalues $\gamma_1,\ldots,\gamma_s$ lie in $\mathbb{Q}\cap(\mathbb{Z}_{p(v)}\setminus\mathbb{Z})$ $(s \leq \nu)$. Let $\nu_1\ldots,\nu_s$ be respectively the multiplicities of $\gamma_1,\ldots,\gamma_s$. Then there exist two positive real numbers c_3, c_4 such that, for any positive integer $n > 0$, we*

have

$$c_3(n+1)^{(1-\nu)^3}|\tau|_v^{n+2}\Big(\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}\Big)^{-(1-\nu)^2}\Big|\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{\nu-2}\le$$

$$\|C_{\Lambda,\tau}(-n)\|_v\le \tag{4.10}$$

$$c_4(n+1)^{(\nu-1)^2}|\tau|_v^{n+2}\Big(\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}\Big)^{\nu-1}\Big|\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{-1}.$$

In particular,

$$\lim_{n\to+\infty}\|C_{\Lambda,\tau}(-n)\|_v^{1/n}=|\tau|_v\pi_v^{-1}.$$

Proof. Lemma (4.1) applies also for $-\Lambda$ instead of Λ since the eigenvalues of $-\Lambda$ belong also to $\mathbb{Q}\cap(\mathbb{Z}_{p(v)}\setminus\mathbb{Z})$. Then, there exist two positive real numbers c_1', c_2', such that, for any integer $n>0$, we have

$$\begin{aligned} c_1'|\tau|_v^{-(n+1)}\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}\le \|C_{-\Lambda,\tau}(n+1)\|_v\le\\ c_2'(n+1)^{\nu-1}|\tau|_v^{-(n+1)}\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}.\end{aligned}\tag{4.11}$$

On the other hand, we have $\det C_{-\Lambda,\tau}(n+1)=\tau^{-(n+1)\nu}\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}$ (see proof of Lemma 4.1), and by (4.2),

$$\Lambda^{-1}C_{-\Lambda,\tau}(n+1)^{-1}=\tau^{-1}C_{\Lambda,\tau}(-n).$$

Applying (4.9) and (4.8) with $Y=\Lambda^{-1}$ and $Z=C_{-\Lambda,\tau}(n+1)^{-1}$, we get

$$\begin{aligned}\|\Lambda^{-1}\|_v\|C_{-\Lambda,\tau}(n+1)\|_v^{-(1-\nu)^2}|\det C_{-\Lambda,\tau}(n+1)|_v^{\nu-2}\le\|\Lambda^{-1}C_{-\Lambda,\tau}(n+1)^{-1}\|_v\le\\ \le\|\Lambda^{-1}\|_v\|C_{-\Lambda,\tau}(n+1)\|_v^{\nu-1}|\det C_{-\Lambda,\tau}(n+1)|_v^{-1}.\end{aligned}\tag{4.12}$$

Replacing now $\det C_{-\Lambda,\tau}(n+1)$ (resp. $\Lambda^{-1}C_{-\Lambda,\tau}(n+1)^{-1}$) with its value in (4.12) and using (4.11), we get

$$\|\Lambda^{-1}\|_v\Big(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|\tau^{-(n+1)}(-\gamma_j+1)_{n+1}|_v\}\Big)^{-(1-\nu)^2}\Big|\tau^{-(n+1)\nu}\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{\nu-2}$$

$$\le\|\tau^{-1}C_{\Lambda,\tau}(-n)\|_v\le$$

$$\|\Lambda^{-1}\|_v\Big(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|\tau^{-(n+1)}(-\gamma_j+1)_{n+1}|_v\}\Big)^{\nu-1}\Big|\tau^{-(n+1)\nu}\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{-1}.$$

or again

$$|\tau|_v^{n+2}\|\Lambda^{-1}\|_v\Big(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}\Big)^{-(1-\nu)^2}\Big|\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{\nu-2}$$

$$\le\|C_{\Lambda,\tau}(-n)\|_v\le$$

$$|\tau|_v^{n+2}\|\Lambda^{-1}\|_v\Big(c_2'(n+1)^{\nu-1}\max_{1\le j\le s}\{|(-\gamma_j+1)_{n+1}|_v\}\Big)^{\nu-1}\Big|\prod_{1\le j\le s}\Big((-\gamma_j+1)_{n+1}\Big)^{\nu_j}\Big|_v^{-1}.$$

Putting now $c_3 = \|\Lambda^{-1}\|_v (c_2')^{-(1-\nu)^2}$ and $c_4 = \|\Lambda^{-1}\|_v (c_2')^{\nu-1}$, we get (4.10). The last statement of Lemma 4.3 follows from (4.10), Lemma 3.6, and the fact: $\nu_1 + \ldots \nu_s = \nu$. $\square$

4.3. Formal Laplace transform in several variables. Let $\underline{\tau} = (\tau_1, \ldots, \tau_d) \in (K \setminus \{0\})^d$ and let $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_d) \in (\mathrm{GL}_\nu(K))^d$ such that the matrices Λ_i mutually commute and such that all the eigenvalues of $\Lambda_1, \ldots, \Lambda_d$ are in $K \setminus \mathbb{Z}$. Put

$$\underline{x}^{\underline{\Lambda}} := x_1^{\Lambda_1} \ldots x_d^{\Lambda_d}, \quad \text{and} \quad \underline{\alpha}\mathbb{I}_\nu := (\alpha_1 \mathbb{I}_\nu, \ldots, \alpha_d \mathbb{I}_\nu).$$

Definition 4.4. For any integer $\mu \geq 1$ and any $\mu \times \nu$ matrix $Y(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}$ with entries in $K[[\underline{x}, 1/\underline{x}]]$, we define the *Laplace transform of* $f := Y(\underline{x})\underline{x}^{\underline{\Lambda}}$ with respect to $\underline{\Lambda}$ and to $\underline{\tau}$ as follows:

(4.13)

$$\begin{aligned}
\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(Y(\underline{x})\underline{x}^{\underline{\Lambda}}) &= \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} \mathcal{L}_{\Lambda_i}^{\tau_i}(x_i^{\Lambda_i + \alpha_i \mathbb{I}_\nu}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) x_i^{-\Lambda_i - (\alpha_i + 1)\mathbb{I}_\nu} \\
&= \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) \underline{x}^{-\underline{\Lambda} - (\underline{\alpha} + \underline{1})\mathbb{I}_\nu} \\
&= \Big(\sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Y_{\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(\alpha_i) \underline{x}^{-\underline{\alpha}\mathbb{I}_\nu} \Big) \underline{x}^{-\underline{\Lambda} - \mathbb{I}_\nu},
\end{aligned}$$

and we set

$$Z_{\underline{\Lambda}, \underline{\alpha}}^{\underline{\tau}} = Y_{-\underline{\alpha}} \prod_{1 \leq i \leq d} C_{\Lambda_i, \tau_i}(-\alpha_i), \quad \text{for all} \quad \underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d,$$

and,

$$Z_{\underline{\Lambda}}^{\underline{\tau}}(\underline{x}) = \sum_{\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d} Z_{\underline{\Lambda}, \underline{\alpha}}^{\underline{\tau}} \underline{x}^{-\underline{\alpha}},$$

so that

$$\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(Y(\underline{x})\underline{x}^{\underline{\Lambda}}) = Z_{\underline{\Lambda}}^{\underline{\tau}}(\underline{x}) \underline{x}^{-\underline{\Lambda} - \mathbb{I}_\nu}.$$

Remark 4.5. The transformation $\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}$ is well defined since, by construction, all the matrices $C_{\Lambda_i, \tau_i}(\alpha_i)$ mutually commute for all $\underline{\alpha}$. In addition, according to formula (4.3), we check easily that this Laplace transform commutes with the derivations in the following sense:

$$\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}\Big(\underline{\partial}^{\underline{\beta}}(f)\Big) = \prod_{1 \leq i \leq d} (\tau_i x_i)^{\beta_i} \mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(f) \ \text{ and } \ \mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}\Big(\prod_{1 \leq i \leq d} x_i^{\beta_i} f\Big) = \frac{(-1)^{|\underline{\beta}|}}{\prod_{1 \leq i \leq d} \tau_i^{\beta_i}} \underline{\partial}^{\underline{\beta}}(\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(f)), \tag{4.14}$$

for any $\underline{\beta} = (\beta_1 \ldots, \beta_d) \in \mathbb{N}^d$. Also, we find

(4.15)

$$\mathcal{L}_{-\underline{\Lambda}}^{\underline{\tau}}\big(\mathcal{L}_{\underline{\Lambda}}^{\underline{\tau}}(Y(\underline{x})\underline{x}^{\underline{\Lambda}})\big) = (-1)^d \prod_{1 \leq i \leq d} \tau_i Y(-\underline{x}) \prod_{1 \leq i \leq d} \Lambda_i^{-1} x^{\underline{\Lambda}}, \quad \text{where} \quad -\underline{x} = (-x_1, \ldots, -x_d).$$

This leads to state

Proposition 4.6. *Assume that $\mu = 1$ and that all the entries of f are solutions of a differential equation $\phi \in K[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$. Then, all the entries of $\mathcal{L}^{\tau}_{\underline{\Lambda}}(f)$ are solutions of $\mathcal{F}_{\underline{\tau}}(\phi)$.*

The formal transformation $\mathcal{L}^{\tau}_{\underline{\Lambda}}$ has moreover the following arithmetic properties:

Proposition 4.7. *Let v be a finite place in Σ_{f}. Under notation of definition 4.4, assume that the matrices $\Lambda_1, \dots, \Lambda_d$ belong to $\mathrm{GL}_\nu(\mathbb{Q})$, all their eigenvalues are in $\mathbb{Q} \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$ and $\tau_1 = \tau_2 \dots = \tau_d = \tau \in K \setminus \{0\}$. Then*

$$\limsup_{|\underline{\alpha}| \to +\infty} \|Z^{\tau}_{\underline{\Lambda},\underline{\alpha}}\|_v^{1/|\underline{\alpha}|} \leq \pi_v^{-1} |\tau|_v \limsup_{|\underline{\alpha}| \to +\infty} \|Y_{-\underline{\alpha}}\|_v^{1/|\underline{\alpha}|},$$

and

$$\limsup_{|\underline{\alpha}| \to +\infty} \|Z^{\tau}_{\underline{\Lambda},-\underline{\alpha}}\|_v^{1/|\underline{\alpha}|} \leq \pi_v |\tau|_v^{-1} \limsup_{|\underline{\alpha}| \to +\infty} \|Y_{\underline{\alpha}}\|_v^{1/|\underline{\alpha}|}.$$

Proof. Let $\gamma_{i,1}, \dots, \gamma_{i,\nu}$ be the eigenvalues of Λ_i for $i = 1, \dots, d$. According to lemmas 4.1 and 4.3, for each $C_{\Lambda_i,\tau}$, there exist two positive constants $c_{2,i}, c_{4,i} > 0$ such that for any $\alpha_i > 0$, we have

$$\|C_{\Lambda_i,\tau}(\alpha_i)\|_v \leq c_{2,i} \alpha_i^{\nu-1} |\tau|_v^{-\alpha_i} \max_{1 \leq j \leq \nu} \{|(\gamma_{i,j}+1)_{\alpha_i}|_v\}.$$

and

$$\|C_{\Lambda_i,\tau}(-\alpha_i)\|_v \leq \\ c_{4,i}(\alpha_i+1)^{(\nu-1)^2} |\tau|_v^{\alpha_i+2} \Big(\max_{1 \leq j \leq \nu} \{|(-\gamma_{i,j}+1)_{\alpha_i+1}|_v\} \Big)^{\nu-1} \Big| \prod_{1 \leq j \leq \nu} (-\gamma_{i,j}+1)_{\alpha_i+1} \Big|_v^{-1}.$$

If we set $\underline{j} = (j_1, \dots, j_d)$, we get

$$\prod_{1 \leq i \leq d} \|C_{\Lambda_i,\tau}(\alpha_i)\|_v \leq \prod_{1 \leq i \leq d} (c_{2,i} \alpha_i^{\nu-1}) |\tau|_v^{-|\underline{\alpha}|} \max_{\underline{j} \leq (\nu,\dots,\nu)} \Big\{ \Big| \prod_{1 \leq i \leq d} (\gamma_{i,j_i}+1)_{\alpha_i} \Big|_v \Big\}.$$

and

$$\prod_{1 \leq i \leq d} \|C_{\Lambda_i,\tau}(-\alpha_i)\|_v \leq \\ \prod_{1 \leq i \leq d} (c_{4,i}(\alpha_i+1)^{(\nu-1)^2}) |\tau|_v^{|\underline{\alpha}|+2d} \Big(\max_{\underline{j} \leq (\nu,\dots,\nu)} \Big\{ \Big| \prod_{1 \leq i \leq d} (-\gamma_{i,j_i}+1)_{\alpha_i+1} \Big|_v \Big\} \Big)^{\nu-1} \Big| \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq \nu}} (-\gamma_{i,j}+1)_{\alpha_i+1} \Big|_v^{-1}.$$

In addition, since all the eigenvalues γ_{i,j_i} lie in $\mathbb{Q} \cap \mathbb{Z}_{p(v)} \setminus \mathbb{Z}$, we have by Lemma 3.6,

$$\limsup_{|\underline{\alpha}| \longrightarrow \infty} \Big| \prod_{1 \leq i \leq d} (-\gamma_{i,j_i}+1)_{\alpha_i+1} \Big|_v^{1/\underline{\alpha}} = \limsup_{|\underline{\alpha}| \longrightarrow \infty} \Big| \prod_{1 \leq i \leq d} (-\gamma_{i,j_i}+1)_{\alpha_i} \Big|_v^{1/\underline{\alpha}} = \pi_v.$$

Combining these observations with the fact $\|Z^{\tau}_{\underline{\Lambda},\underline{\alpha}}\|_v \leq \|Y_{-\underline{\alpha}}\|_v \prod_{1 \leq i \leq d} \|C_{\Lambda_i,\tau_i}(-\alpha_i)\|_v$ for any $\underline{\alpha} \in \mathbb{N}^d \cup (-\mathbb{N})^d$, we get the proposition. $\square$

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